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P3_2 Terminal velocities due to relativistic Doppler effects

R.H. Peck, H.W. Buttery, C.D.Y. Moore, C.J. Middleton

Department of Physics and Astronomy, University of Leicester, Leicester, LE1 7RH

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Abstract

We find the speed of a spacecraft at which the radiation pressure from blue-shifted starlight ahead is enough to balance the propulsive force of its engines. We find that the drag pressure obeys Eq.(6), calculate the terminal velocity of the *Project Daedalus* 2nd stage to be $(100 - 4.6 \times 10^{-19})\%c$, and use Eq.(6) to find that on the Daedalus only 1.1×10^{-15} N of drag results from this effect at the planned $0.12c$ cruise speed.

Introduction

Starlight as seen from a relativistic craft will appear blue-shifted and concentrated ahead [1]. This increases its intensity and allows it to generate greater radiation pressures. As this pressure will increase with speed we realise that a spacecraft of frontal area, A, with given thrust, F, has a terminal velocity, V_t. We model the craft travelling through a uniform field of stars [2] with a luminosity of 3.8×10^{26} W, giving a 1.9×10^{-10} Wm^{-2} [2] flux, hence an intensity as seen from rest of $I_0 = 1.5 \times 10^{-11}$ Wm^{-2}sr^{-1}. Being uniform, the radiation field seen from the craft is not altered by relativistic length contraction [3]. We assume a perfectly reflective surface on the vehicle's frontal area. To keep the paper to an appropriate length we ignore general relativity by not modelling any further blue-shifting of incoming light that may be caused by general relativity if the craft's relativistic mass increase [3] generates a significant gravity-well.

Radiation pressure

Radiation pressure, P, can be calculated by integrating the intensity of a radiation field using Eq.(1) [4].

$$P = \frac{2}{c}\int_0^{2\pi} d\phi \int_0^{\pi/2} I_{obs}(\theta_{obs}) \cos^2\theta_{obs} \sin\theta_{obs} d\theta_{obs} \tag{1}$$

Where c is the speed of light, I_{obs} is the intensity of light observed onboard the craft, and ϕ and θ_{obs} are the "longitude" and "co-latitude" (respectively) if the craft's velocity vector is considered as the "north pole" of a sphere. The factor of 2 indicates a reflective surface. The limits are chosen such that only the frontal hemisphere around the craft is considered; it is assumed that the radiation pressure from behind the craft can be neglected as light from behind is red-shifted and vastly reduced in intensity.

Headlamp effect

We begin by considering the Headlamp effect. In Eq.(1) the "co-latitude" is given in the frame of the relativistic craft. However, the light seen at θ_{obs} in the moving craft's frame will appear to come from θ_0 in the frame of the universe. The different angle seen from aboard the craft is a relativistic equivalent to a classical phenomenon: raindrops that fall vertically appear to fall at an angle when viewed from a moving vehicle. Eq.(2) [1] shows that at relativistic speeds much of the field of view is compressed into a narrow cone ahead.

$$\cos\theta_{obs} = \frac{\cos\theta_0 + \frac{V}{c}}{1 + \frac{V}{c}\cos\theta_0} \tag{2}$$

V is the craft's velocity, defined here as positive when the craft is moving towards the light source, whereas [1] uses a "$V > 0$ is recession" convention (hence different signs).

Relativistic Doppler effects

The Doppler effect causes the wavelength of radiation to appear to change due to the relative motion of the source and observer. When the observer and source are approaching one another the wavelength is shortened. At relativistic speeds the situation is made more complex by the time dilation experienced. The relativistic Doppler effect is given by Eq.(3) [1].

$$\frac{\nu_{obs}}{\nu_0} = \frac{1 + \frac{V}{c}\cos\theta_0}{\sqrt{1 - \frac{V^2}{c^2}}} \tag{3}$$

As before, positive V is motion towards the light source, ν_0 is the frequency seen from the stationary emitter's frame, and ν_{obs} is the frequency measured aboard the craft.

An expression for pressure

The relation below (Eq.(4)) has been proven previously [5], where the total intensity, I, has units ($\mathrm{Wm^{-2}sr^{-1}}$), and ν is the light's frequency.

$$\frac{I}{\nu^4} = constant \tag{4}$$

We can now begin substituting into Eq.(1) to calculate the pressure. $I(\theta_{obs})$ can be found by rearranging Eq.(4) to give $I_{obs} = I_0\left(\frac{\nu_{obs}}{\nu_0}\right)^4$. Substituting in Eq.(3) and using a rearranged Eq.(2) to convert the angle to the craft's frame we find that I_{obs} is described by Eq.(5).

$$I_{obs} = I_0\left(\frac{1 + (V/c)\left[\frac{\cos\theta_{obs} - (V/c)}{1 - (V/c)\cos\theta_{obs}}\right]}{\sqrt{1 - (V/c)^2}}\right)^4 \tag{5}$$

Substituting Eq.(5) into Eq.(1) gives Eq.(6).

$$P = \frac{4\pi}{3}\frac{\left(\frac{V}{c} + 1\right)^2 I_0}{c\left(1 - \frac{V}{c}\right)} \tag{6}$$

Terminal velocity

At V_t the thrust equals the drag. Solving $F = AP$ for V/c, with P from Eq.(6), we get Eq.(7).

$$\left(\frac{V_t}{c}\right) = \frac{1}{2}\left(-\frac{3Fc}{4AI_0\pi} - 2 \pm \sqrt{\frac{3Fc}{4AI_0\pi}}\sqrt{\frac{3Fc}{4AI_0\pi} + 8}\right) \tag{7}$$

We can ignore the negative solution and hence calculate V_t for the Daedalus craft [6]. The Daedalus probe 2nd stage would have a roughly circular frontal cross-section of $A = 3630\ \mathrm{m}^2$ (found from the scale-diagram [7]) and a 2nd stage engine with thrust, $F = 663$ kN [6]. Using values from the introduction and above, Eq.(7) gives a terminal velocity of $(100 - 4.6 \times 10^{-19})\%c$. The Daedalus' planned cruise speed is $0.12c$ [6], using Eq.(6) and A we find that a drag of only 1.1×10^{-15} N is experienced at this speed.

Conclusion

The cruise speed of Daedalus is $0.12c$, well below V_t of $(100 - 4.6 \times 10^{-19})\%c$. At $0.12c$ the drag is 1.1×10^{-15} N. We conclude that these drag effects are negligible for an interstellar mission.

References

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